Completing the Square to Find Turning Points

Lesson Objective

To be able to complete the square in a quadratic expression in order to find the turning points of the graph of the function.

Success Criteria

- To be able to complete the square where the coefficient of x squared is 1
- To be able to complete the square where the coefficient of x squared is not equal to 1
- To identify the position and nature of the turning point

Starter

Practise expanding an expression of the form $(x+a)^2$ for different values of a.

Your teacher will give you some different values for a!

Quadratic Expressions

A quadratic expression is one in which the maximum power of the unknown is 2, for example:

- m^2+2m-7

or

- x^2+3x

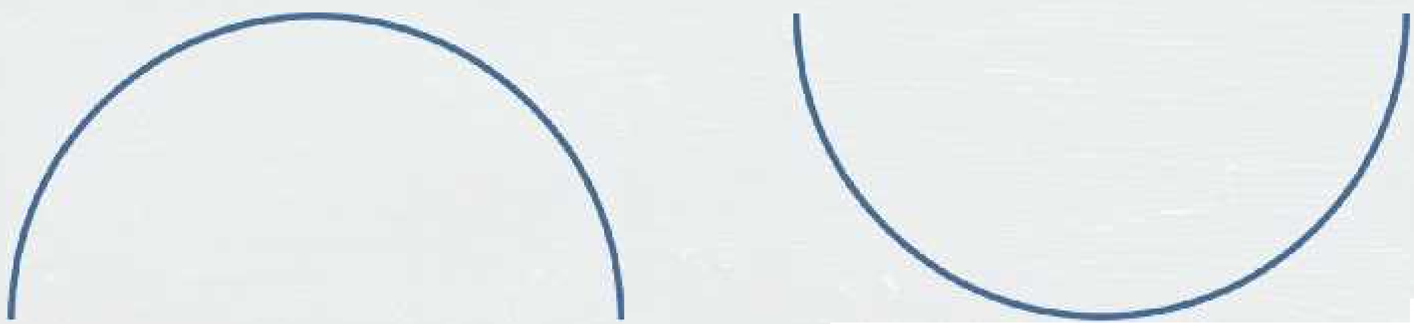
or

- $-3a^2+4$

The Two Types of Quadratic Graph

When we draw a quadratic graph (for example, $y=x^2+4x-3$), the shape of the graph is called a parabola.

Parabolas look like this, depending on whether the coefficient of x^2 (the number of x^2s in the expression) is negative or positive:



Negative parabola

Positive parabola

Positive and Negative Coefficients

In the equation $y=5x^2+4x-3$, the coefficient of x^2 is 5. In the equation $y=-x^2+3x-5$, the coefficient of x^2 is -1.

When the coefficient of x^2 is positive, the graph of y will be like the parabola on the right, like a smile, and will have a minimum value or turning point, as indicated.

When the coefficient of x^2 is negative, the graph of y will be like the one on the left, like a frown, and will have a maximum value or turning point.

Negative parabola

Positive parabola

Finding Out More About the Turning Point

We can find out more about the turning point of a quadratic graph by ‘completing the square’.

When the Coefficient of x^2 is 1

When the coefficient of x^2 is 1, completing the square involves writing the expression in x in this form:

$(x+r)^2+s$, where r and s are constant terms.

In the equation $y=x^2+4x-3$, we begin by halving the coefficient of x to find r.

Half of 4 is 2.

We now have $y=(x+2)^2+s$

Adjusting for the Constant

Notice that expanding $(x+2)^2$ gives x^2+4x+4. It has given us the x^2+4x, that we needed, which is good (that's why we chose to put a 2 in the brackets), but has also given us a 4 – which we didn't need. We must subtract 4 before we write -3 after our brackets.

This gives $y=(x+2)^2-4-3$

Which gives $y=(x+2)^2-7$

We have completed the square.

Another Example

Let's have another go, this time with $y=x^2-10x+30$.

Again, we begin by halving the coefficient of x. Half of −10 is −5, which will be our value for r.

We now have $y=(x-5)^2+s$

Expanding $(x-5)^2$ gives $x^2-10x+25$. We got the x^2-10x that we wanted, but must again compensate for the unnecessary 25 by subtracting 25 before we add on the constant term 30. This gives:

$y=(x-5)^2-25+30$

Which gives $y=(x-5)^2+5$

When the Coefficient of x^2 is not 1

When the coefficient of x^2 is not 1, completing the square involves writing the expression in x in this form:

$q(x+r)^2+s$, where q, r and s are constant terms.

Q **R** **S**

An Example $y=4x^2+24x+3$

Let's begin by taking a factor of 4 out of the first two terms, as that will put us in a position where we can focus again on a single x^2.

This gives $y=4(x^2+6x)+3$

Now, we will focus on the expression within the brackets. Begin again by halving the coefficient of x; half of 6 is 3. 3 will be the value of r in our completed equation.

But notice that $(x+3)^2$, when expanded, gives us x^2+6x+9. We needed the first two terms in this, but the 9 is surplus to requirements.

$(x+3)^2-9$ gives us the x^2+6x which we were looking for, so we will use it in

$y=4(x^2+6x)+3$ as follows:

$y=4[(x+3)^2-9]+3$

Multiplying the square term and the 9 in the bracket by the 4 gives:

$y=4(x+3)^2-36+3$

or $y=4(x+3)^2-33$

Another Example $y=2x^2+5x+9$

Now try $y=2x^2+5x+9$

Taking the factor of 2 from the first two terms:

$y=2(x^2+\frac{5}{2}x)+9$

Half of $\frac{5}{2}$ is $\frac{5}{4}$.

$(x+\frac{5}{4})^2$ will give us the $x^2+\frac{5}{2}x$ that we need, but also a $(\frac{5}{4})^2=\frac{25}{16}$ that we did not need.

In that case:

$y=2[(x+\frac{5}{4})^2-\frac{25}{16}]+9$

$y=2(x+\frac{5}{4})^2-\frac{25}{8}+9$

$y=2(x+\frac{5}{4})^2+5\frac{7}{8}$

One More Example $y=-x^2+6x-3$

Now try $y=-x^2+6x-3$

This time, the coefficient of x^2 is -1, so we take a factor of -1 out of the first 2 terms:

$y=-(x^2-6x)-3$

Half of -6 is -3, so we begin with -3 in the bracket with the x.

$(x-3)^2$ gives x^2-6x+9 when expanded, a $-6x$ that we needed and a 9 that we did not need.

$y=-[(x-3)^2-9]-3$

$y=-(x-3)^2+9-3$

$y=-(x-3)^2+6$

Finding the Minimum or Maximum Point

We will begin with $y=x^2+4x-3$

Completing the square gave: $y=(x+2)^2-7$

Since the coefficient of x^2 is 1, and that is a positive number, we know that the parabola will look like a smile and will have a minimum point.

If we study the equation $y=(x+2)^2-7$, we can find out what the minimum value of y is. Squaring any number, positive or negative, gives a number greater than or equal to zero. We know therefore that $(x+2)^2$ can never be less than zero, but it will be zero, when $x=-2$. Since we're looking for the minimum value of y, we want $(x+2)^2$ to be as low as possible, that is 0. When $(x+2)^2=0$, $y=0-7$, that is -7. The minimum point therefore is $(-2,-7)$.

$y=x^2-10x+30$ gave $y=(x-5)^2+5$

Again, we have a positive coefficient of x^2, that is 1, so the parabola will take the shape of a smile. When $x=5$, we get a minimum value of $(x-5)^2$, and that minimum value is 0. That gives $y=0+5=5$. This time, the minimum point is (5,5).

$y=4x^2+24x+3$ gave $y=4(x+3)^2-33$

There is a positive coefficient of x^2 in this expression. The curve will take the shape of a smile and will have a minimum point. Again $(x+3)^2$ has a minimum value of 0. Multiplying it by 4 still gives 0 and the minimum value for y would be $0-33$. The minimum point is at $(-3, -33)$.

$y=-x^2+6x-3$ gave $y=-(x-3)^2+6$

Since this equation has a negative coefficient of x^2, the parabola is a frown shape this time, which means that it will have a maximum turning point. $-(x-3)^2$, will never be positive, since $(x-3)^2$ is the result of squaring a number and will never be negative. The biggest value of $-(x-3)^2$ is therefore 0, and occurs when $x=3$. That means that the maximum value of y is $0+6=6$. The maximum point is (3,6).

The General Case

Generally, if the equation of the curve is $y=q(x+r)^2+s$ after the square has been completed, then:

- when q is positive, the minimum point is (–r,s);
- when q is negative, the maximum point is (–r,s).

Your Turn!

Have a go at the Completing the Square activity sheet.

How many can you get correct?

$y=q(x+r)^2+s$

When we have completed the square in a quadratic function, the equation will look like this:

$y=q(x+r)^2+s$

And we can deduce from this that:

if q is positive, the curve looks like a smile and has a minimum point at $(-r,s)$

if q is negative, the curve looks like a smile and has a maximum point at $(-r,s)$

Question 1

State whether the turning point in the following curves are a maximum or minimum and give their coordinates:

$y=(x-5)^2+6$

$y=-2(x+7)^2-9$

Minimum: (5, 6) Maximum: (-7, -9)

Question 2

The equation of a curve is $y=2(x+a)^2+b$.

If the curve has a minimum point at (7,−3), state the values of a and b.

(-7, -3)

Question 3

The curve $y=(x+7)^2-3$ is translated by the vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

What is the equation of the new curve

a) written in the completed square form?

b) written in the form $y=ax^2+bx+c$?

a) $y=(x+5)^2-8$ b) $y=x^2+10x+17$

Completing the Square to Find Turning Points

For each of the following quadratic graphs:

a. State whether it has a maximum or a minimum turning point.

b. Complete the square.

c. Give the coordinates of the turning point.

1. $y = x^2 + 10x + 2$ (worked example)

   a. The coefficient of x^2 is positive so the graph has a minimum point.

   b. The coefficient of x is 10; half of 10 is 5, so we place 5 in the brackets along with the x: $(x + 5)$
   $(x + 5)^2$ expands to give $x^2 + 10x + 25$, which means that we need to subtract 25:
   $y = (x + 5)^2 - 25 + 2$
   $y = (x + 5)^2 - 23$

   c. y is minimum when $x = -5$, and this gives a y value of -23. The turning point is $(-5, -23)$.

2. $y = x^2 - 8x + 3$

   a.

   b.

   c.

3. $y = x^2 + 9x - 9$

   a.

   b.

   c.

4. $y = x^2 - 3x - 6$

   a.

   b.

   c.

5. $y = 2x^2 + 18x - 23$

   a.

   b.

   c.

6. $y = 3x^2 + 15x + 12$

   a.

   b.

   c.

7. $y = 5x^2 - 4x + 2$

   a.

   b.

   c.

8. $y = 2x^2 - 9x + 11$

   a.

   b.

   c.

9. $y = 4x^2 + 6x + 12$

   a.

   b.

   c.

10. $y = 5x^2 + 3x - 9$

    a.

    b.

    c.

11. $y = -x^2 + 10x + 5$

a.

b.

c.

12. $y = 7x^2 - 3x + 12$

a.

b.

c.

13. $y = -x^2 - 4x - 11$

a.

b.

c.

14. $y = -x^2 + 8x + 7$

a.

b.

c.

15. $y = 4x^2 + 4x + 2$

a.

b.

c.

16. $y = -3x^2 + 4x + 9$

a.

b.

c.

Completing the Square to Find Turning Points **Answers**

2. $y = x^2 - 8x + 3$
   a. minimum
   b. $y = (x - 4)^2 - 13$
   c. $(4, -13)$

3. $y = x^2 + 9x - 9$
   a. minimum
   b. $y = (x + \frac{9}{2})^2 - \frac{117}{4}$
   c. $(\frac{9}{2}, -\frac{117}{4})$

4. $y = x^2 - 3x - 6$
   a. minimum
   b. $y = (x - \frac{3}{2})^2 - \frac{33}{4}$
   c. $(\frac{3}{2}, \frac{-33}{4})$

5. $y = 2x^2 + 18x - 23$
   a. minimum
   b. $y = 2(x + \frac{9}{2})^2 - \frac{127}{2}$
   c. $(-\frac{9}{2}, -\frac{127}{2})$

6. $y = 3x^2 + 15x + 12$
   a. minimum
   b. $y = 3(x + \frac{5}{2})^2 - \frac{27}{4}$
   c. $(-\frac{5}{2}, -\frac{27}{4})$

7. $y = 5x^2 - 4x + 2$
   a. minimum
   b. $y = 5(x - \frac{2}{5})^2 + \frac{6}{5}$
   c. $(\frac{2}{5}, \frac{6}{5})$

8. $y = 2x^2 - 9x + 11$
   a. minimum
   b. $y = 2(x - \frac{9}{4})^2 + \frac{7}{8}$
   c. $(\frac{9}{4}, \frac{7}{8})$

9. $y = 4x^2 + 6x + 12$
   a. minimum
   b. $y = 4(x + \frac{3}{4})^2 + \frac{39}{4}$
   c. $(-\frac{3}{4}, \frac{39}{4})$

10. $y = 5x^2 + 3x - 9$
    a. minimum
    b. $y = 5(x + \frac{3}{10})^2 - \frac{189}{20}$
    c. $(-\frac{3}{10}, -\frac{189}{20})$

11. $y = -x^2 + 10x + 5$
    a. maximum
    b. $y = -(x - 5)^2 + 30$
    c. $(5, 30)$

12. $y = 7x^2 - 3x + 12$
    a. minimum
    b. $y = 7(x - \frac{3}{14})^2 + \frac{327}{28}$
    c. $(\frac{3}{14}, \frac{327}{28})$

13. $y = -x^2 - 4x - 11$
    a. maximum
    b. $y = -(x + 2)^2 - 7$
    c. $(-2, -7)$

14. $y = -x^2 + 8x + 7$
    a. maximum
    b. $y = -(x - 4)^2 + 23$
    c. $(4, 23)$

15. $y = 4x^2 + 4x + 2$
    a. minimum
    b. $y = 4(x + \frac{1}{2})^2 + 1$
    c. $(-\frac{1}{2}, 1)$

16. $y = -3x^2 + 4x + 9$
    a. maximum
    b. $y = -3(x - \frac{2}{3})^2 + \frac{31}{3}$
    c. $(\frac{2}{3}, \frac{31}{3})$

Completing the Square to Find Turning Points
Teaching Ideas

Lesson Objective:	To be able to complete the square in a quadratic expression in order to find the turning points of the graph of the function.
Success Criteria:	• To be able to complete the square where the coefficient of x squared is 1 • To be able to complete the square where the coefficient of x squared is not equal to 1 • To identify the position and nature of the turning point
Context:	This is a stand-alone lesson, which may be taught during a topic on graphs.

Starter

Ask pupils to practise expanding an expression of the form $(x+a)^2$, for various values of a: negative, fractional and positive. You could simply write $(x+a)^2$ on the board several times with different values for a and give pupils time to work out the answers in their books.

Main Activities

Completing the Square to Find Turning Points

Work through the **PowerPoint Presentation Completing the Square to Find Turning Points** with pupils. The presentation explains the two types of quadratic graph and positive and negative coefficients. It then explains – with examples – how to complete the square when the coefficient of x^2 is 1 and when it is not 1. Work through the examples slowly and carefully – you could ask pupils to have a go at these independently or in pairs before giving them the answers and showing them how to work them out. Finally, there is a section on how to find the minimum or maximum point, again with plenty of examples for you to use to test pupils' understanding and let them practise.

Completing the Square Activity Sheet

Give pupils the **Completing the Square to Find Turning Points activity sheet**. Answers are also available.

Plenary

The plenary gives some more questions that can be answered as a class, in pairs or independently. Answers are also provided on the following slides.